

ON THE EXISTENCE OF SMOOTH SOLUTIONS FOR FULLY NONLINEAR PARABOLIC EQUATIONS WITH MEASURABLE “COEFFICIENTS” WITHOUT CONVEXITY ASSUMPTIONS

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ABSTRACT. We show that for any uniformly parabolic fully nonlinear second-order equation with bounded measurable “coefficients” and bounded “free” term in any cylindrical smooth domain with smooth boundary data one can find an approximating equation which has a unique continuous solution with the first derivatives bounded and the second spacial derivatives locally bounded. The approximating equation is constructed in such a way that it modifies the original one only for large values of the unknown function and its spacial derivatives.

1. INTRODUCTION AND MAIN RESULT

In this article, we consider parabolic equations

$$\partial_t v(t, x) + H[v](t, x) := \partial_t v(t, x) + H(v(t, x), Dv(t, x), D^2 v(t, x), t, x) = 0 \quad (1.1)$$

in subdomains of $\mathbb{R}^{d+1} = \mathbb{R} \times \mathbb{R}^d$, where

$$\mathbb{R}^d = \{x = (x_1, \dots, x_d) : x_1, \dots, x_d \in \mathbb{R}\}.$$

Here

$$\partial_t = \partial/\partial t, \quad D^2 u = (D_{ij} u), \quad Du = (D_i u), \quad D_i = \frac{\partial}{\partial x_i}, \quad D_{ij} = D_i D_j.$$

We prove that for any uniformly parabolic fully nonlinear second-order equation with bounded measurable “coefficients” and bounded “free” term in a given cylindrical smooth domain with smooth boundary data, one can find an approximating equation which has a unique continuous solution with the first derivatives bounded and the second spacial derivatives locally bounded. The novelty of our result is that we do not impose any convexity assumptions on the equation. This is a continuation of [13], in which a similar result was obtained for elliptic equations.

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The convexity of operators plays an important role in the regularity theory of fully nonlinear elliptic and parabolic equations. For elliptic equations without convexity assumptions, the best result one can get is that viscosity solutions are in $C^{1+\alpha}$ (see Trudinger [17]) under the condition that the operators are sufficient regular (Hölder) with respect to the independent variables. In fact, N. Nadirashvili and S. Vlăduț [16] found an example which shows that even for elliptic operators independent of the space variables viscosity solutions may not have bounded second-order derivatives. For equations with measurable coefficients, M. G. Crandall, M. Kocan, and A. Świąch [4] developed a theory of L_p -viscosity solutions (see also the references therein).

Interior W_p^2 a priori estimates for elliptic equations was first derived by L. Caffarelli under an assumption that certain estimates hold for equations with zero “free” term, which are known to hold only for H that are either convex or concave with respect to v , Dv , and D^2v (see [1] and [2]). Note that some particular cases of $C^{2+\alpha}$ a priori estimates without this assumption can be found in [3] and [7]. This line of research was continued by L. Wang in [18] who obtained similar interior a priori estimates for parabolic equations, by M. G. Crandall, M. Kocan, and A. Świąch [4] who established the *solvability* in local Sobolev spaces of the boundary-value problems for fully nonlinear parabolic equations, and by N. Winter [19] who established the solvability in the global W_p^2 -space of the associated boundary-value problem in the elliptic case. In the existence parts in [4] and [19] the function H is supposed to be convex with respect to D^2v and continuous in x (concerning the latter assumption see [19, Remark 2.3], [9], and [4, Example 8.3]). It is worth noting that in the above references the authors considered equations like (1.1) with the right-hand side which is not zero but rather a function from an L_p -space. In our setting we can only treat bounded right-hand sides.

In two recent papers [9, 6] the authors used a very different approach to study the W_p^2 theory of fully nonlinear elliptic and parabolic equations with VMO “coefficients”. The convexity of H with respect to D^2v is relaxed for the a priori estimates, but is still assumed in the proof of the existence result. Nevertheless, it is conjectured in [6] that the convexity condition can be dropped or at least relaxed for the existence result.

This conjecture was addressed in [13] and [14]. In [13] the author considered uniformly elliptic fully nonlinear second-order equation of the form $H[v] = 0$ with bounded measurable “coefficients” and bounded “free” term in a given smooth domain with smooth boundary data. It is shown that one can find an approximating equation

$$\max(H[v], P[v] - K) = 0,$$

which has a unique continuous solution with locally bounded second-order derivatives. The approximating equation differs from the original one only for large values of the unknown function and its derivatives. By using this result, in [14] the author established the existence and uniqueness of solutions of fully nonlinear elliptic second-order equations in smooth domains, under

a relaxed convexity assumption with respect to D^2v and a VMO condition with respect to x which are imposed only for large $|D^2v|$.

Roughly speaking, the main idea of [13] is that on the set, say Γ , where the second-order derivatives of v are large we have $P[v] = K$ and in the spirit of the maximum principle the second order derivative on Γ are controlled by their values on the boundary of Γ , where they are under control by the definition of Γ . The implementation of this idea, however, requires sufficient regularity of solutions to (1.2). Since this is not known a priori, the above idea is applied at the level of finite differences.

In this article, we extend the result of [13] to parabolic equations. To state our main results, we introduce a few notation and assumptions. Let \mathcal{S} be the set of symmetric $d \times d$ matrices, fix a constant $\delta \in (0, 1]$, and set

$$\mathcal{S}_\delta = \{a \in \mathcal{S} : \delta|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \delta^{-1}|\xi|^2, \quad \forall \xi \in \mathbb{R}^d\},$$

where and everywhere in the article the summation convention is enforced unless specifically stated otherwise.

Assumption 1.1. (i) The function $H(u, t, x)$, $u = (u', u'')$,

$$u' = (u'_0, u'_1, \dots, u'_d) \in \mathbb{R}^{d+1}, \quad u'' \in \mathcal{S}, \quad (t, x) \in \mathbb{R}^{d+1},$$

is measurable with respect to (t, x) for any u , and Lipschitz continuous in u for every $(t, x) \in \mathbb{R}^{d+1}$.

(ii) For any (t, x) , at all points of differentiability of $H(u, t, x)$ with respect to u , we have

$$(H_{u''_{ij}}) \in \mathcal{S}_\delta, \quad |H_{u'_k}| \leq \delta^{-1}, \quad k = 1, \dots, d, \quad 0 \leq -H_{u'_0} \leq \delta^{-1}.$$

(iii) Finally,

$$\bar{H} := \sup_{(t,x) \in \mathbb{R}^{d+1}} |H(0, t, x)| < \infty.$$

Remark 1.2. It is almost obvious that Assumption 1.1 (ii) is equivalent to the requirement that, for any $u \in \mathbb{R}^{d+1} \times \mathcal{S}$, $x, \xi \in \mathbb{R}^d$, $\eta \in \{\pm e_1, \dots, \pm e_d\}$, where e_1, \dots, e_d is the set of standard basis vectors in \mathbb{R}^d , and $r \geq 0$, we have

$$\delta|\xi|^2 \leq H(u', u'' + \xi\xi^*, t, x) - H(u', u'', t, x) \leq \delta^{-1}|\xi|^2,$$

$$|H(u' + r(0, \eta), u'', t, x) - H(u', u'', t, x)| \leq \delta^{-1}r,$$

$$H(u', u'', t, x) - \delta^{-1}r \leq H(u' + r(1, 0), u'', t, x) \leq H(u', u'', t, x),$$

where $(0, \eta) = (0, \eta_1, \dots, \eta_d)$ and $(1, 0) = (1, 0, \dots, 0)$.

Let Ω be an open bounded subset of \mathbb{R}^d with C^2 boundary and $-\infty \leq S < T < \infty$. We denote the parabolic boundary of the cylinder $(S, T) \times \Omega$ by

$$\partial'((S, T) \times \Omega) = (\{T\} \times \Omega) \cup ((S, T] \times \partial\Omega).$$

For any $T > 0$, we define $\Omega_T = (0, T) \times \Omega$.

We use the Hölder spaces $C^{\alpha, \beta}$, $\alpha, \beta \in (0, 1]$, of functions of (t, x) which are the spaces of bounded functions having finite Hölder norm of order α in t and β in x . The symbol $C^{1,2}$ stands for the space of bounded functions u for

which $\partial_t u$, Du , and D^2u are bounded and continuous with respect to (t, x) . These spaces are provided with natural norms. We denote by $W_p^{1,2}(\Omega_T)$ the space of functions v defined on Ω_T such that v , Dv , D^2v , and $\partial_t v$ are in $L_p(\Omega_T)$.

Theorem 1.3. *Let $T > 0$ and $K \geq 0$ be fixed constants, and $g \in C^{1,2}(\bar{\Omega}_T)$. There is a constant $\hat{\delta} \in (0, \delta]$ depending only on δ and d and there exists a function $P(u)$ (independent of t, x), satisfying Assumption 1.1 with $\hat{\delta}$ in place of δ , such that the equation*

$$\partial_t v + \max(H[v], P[v] - K) = 0 \quad (1.2)$$

in Ω_T (a.e.) with terminal-boundary condition $v = g$ on $\partial'\Omega$ has a unique solution $v \in C^{1,1}(\bar{\Omega}_T) \cap W_{\infty,loc}^{1,2}(\Omega_T)$. In addition, for all i, j , and $p \in (d + 1, \infty)$,

$$|v|, |D_i v|, \rho |D_{ij} v|, |\partial_t v| \leq N(\bar{H} + K + \|g\|_{C^{1,2}(\Omega_T)}) \quad \text{in } \Omega_T \quad (\text{a.e.}), \quad (1.3)$$

$$\|v\|_{W_p^{1,2}(\Omega_T)} \leq N_p(\bar{H} + K + \|g\|_{W_p^{1,2}(\Omega_T)}), \quad (1.4)$$

$$\|v\|_{C^{\alpha/2,\alpha}(\Omega_T)} \leq N(\|H[0]\|_{L_{d+1}(\Omega_T)} + \|g\|_{C^{\alpha/2,\alpha}(\Omega_T)}), \quad (1.5)$$

where

$$\rho = \rho(x) = \text{dist}(x, \mathbb{R}^d \setminus \Omega),$$

$\alpha \in (0, 1)$ is a constant depending only on d and δ , N is a constant depending only on Ω and δ , whereas N_p only depends on Ω , T , δ , and p .

Finally, $P(u)$ is constructed on the sole basis of δ and d , it is positive homogeneous of degree one and convex in u .

In the proof of Theorem 1.3, we adapt the aforementioned idea in [13] to the parabolic setting. As there, we start at the level of finite differences. Although it is tempting to discretize the equation with respect to both t and x , it turns out that it suffices for us to discretize only with respect to x , so that the discretized equation is a system of ordinary differential equations with respect to t . The estimates of the solution to the discretized equation as well as its first-order space finite differences follow the line in [13] by using a version of the maximum principle in “non-cylindrical” domains; cf. Lemma 4.2. The estimates of the second-order space finite differences are more involved. In order to get their lower bound, we apply Bernstein’s method to the discretized equation. In contrast to the elliptic case, for the upper bound we first need to control the time derivative of the solution, using again Lemma 4.2. The upper bound of the second-order space finite differences is then deduced from the above estimates and the equation itself.

Remark 1.4. Estimate (1.5) follows from other assertions of Theorem 1.3 and the classical results about linear equations with measurable coefficients (see, for instance, Section VII.9 of [15]). Indeed, as is easy to see for $v \in W_p^{1,2}(\Omega_T)$ satisfying (1.2) we have that

$$-\max(H[0], P[0] - K) = \max(H[v], P[v] - K) - \max(H[0], P[0] - K)$$

$$= a_{ij}D_{ij}v + b_iD_iv - cv$$

with some functions $a = (a_{ij}) \in \mathcal{S}_{\hat{\delta}}$, $|b_i| \leq \hat{\delta}^{-1}$, $0 \leq c \leq \hat{\delta}^{-1}$ (cf. the proof of Lemma 2.2). Furthermore,

$$|\max(H[0], P[0] - K)| = |\max(H[0], -K)| \leq |H[0]|.$$

The assertion of Theorem 1.3 concerning uniqueness in our class of functions is also a classical result derived from the parabolic Alexandrov estimate.

Remark 1.5. Even though quite a few auxiliary results from [13] are used in the present article, the main result of [13] is not. It even turns out that it can be derived from Theorem 1.3 and the results of [6]. Of course, such an indirect derivation is somewhat longer than the one given in [13] but yet it is worth mentioning.

Thus, we assume that H and g are independent of t . The proof of the elliptic counterparts of (1.4) and (1.5) consists of just a repetition of the arguments of the present article (using [6]). In what concerns existence and estimate (1.3), we denote by v_T the solution from Theorem 1.3. By (1.3), for any $S \geq 0$ the family v_T , $T \geq S$, is equi-bounded and equi-continuous on Ω_S . It follows that there is a sequence $T(n) \rightarrow \infty$ as $n \rightarrow \infty$ such that $v_{T(n)}$ converge uniformly on each Ω_S to a function v obviously satisfying (1.3) on Ω_∞ . The rules of passing to the limit in fully nonlinear equations (see, for instance, Theorem 3.5.9 of [8]) show that v satisfies (1.2) in Ω_∞ . Since the functions g , H , and P are independent of t , $v(t + T, x)$ satisfies the same equation for any fixed $T \geq 0$ and by uniqueness $v(t, x) = v(t + T, x)$. This means that $v(t, x) = v(x)$, equation (1.2) becomes elliptic, and we obtain all assertions of Theorem 1.1 of [13].

To conclude our comments about Theorem 1.3 we show how P is constructed. By Theorems 3.1 of [10] there exists a set

$$\{l_1, \dots, l_m\} \subset \mathbb{Z}^d,$$

$m = m(\delta, d) \geq d$, chosen on the sole basis of knowing δ and d and there exists a constant

$$\hat{\delta} = \hat{\delta}(\delta, d) \in (0, \delta/4]$$

such that:

(i) We have

$$e_i, e_i \pm e_j \in \{l_1, \dots, l_m\} = \{-l_1, \dots, -l_m\}$$

for all $i, j = 1, \dots, d$ (recall that e_1, \dots, e_d is the standard orthonormal basis of \mathbb{R}^d);

(ii) There exist real-analytic functions $\lambda_1(a), \dots, \lambda_m(a)$ on $\mathcal{S}_{\delta/4}$ such that for any $a \in \mathcal{S}_{\delta/4}$

$$a \equiv \sum_{k=1}^m \lambda_k(a) l_k l_k^*, \quad \hat{\delta}^{-1} \geq \lambda_k(a) \geq \hat{\delta}, \quad \forall k.$$

Now introduce

$$\mathcal{P}(z) = \max_{\substack{\delta/2 \leq a_k \leq 2\delta^{-1} \\ k=1, \dots, m}} \max_{\substack{|b_k| \leq 2\delta^{-1} \\ k=1, \dots, d}} \max_{\delta/2 \leq c \leq 2\delta^{-1}} \left[\sum_{k=1}^m a_k z_k'' + \sum_{k=1}^d b_k z_k' - cz_0' \right],$$

and for $u = (u', u'') \in \mathbb{R}^{d+1} \times \mathcal{S}$ define

$$P(u', u'') = \mathcal{P}(u', \langle u'' l_1, l_1 \rangle, \dots, \langle u'' l_m, l_m \rangle),$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in \mathbb{R}^d .

The remaining part the article is organized as follows. Sections 2 and 3 are devoted to the reduction of proving Theorem 1.3 to proving Theorem 3.2, that is a special case of Theorem 1.3 but under additional assumptions. In Section 4 we consider finite-difference approximations for equations with “constant” coefficients and prove interior estimates for the second-order differences of solutions. In Section 5 by using the results of the previous section we prove an analog of Theorem 1.3 for H , that, as far as the dependence on $D^2 v$ is concerned, include only *pure* second-order derivatives. We complete the proof of Theorem 3.2 in Section 6.

2. REDUCING THEOREM 1.3 TO A PARTICULAR CASE WHERE $-H_{u'_0} \geq \delta$

Suppose that Theorem 1.3 is true under the additional assumption that

$$-H_{u'_0} \geq \delta \tag{2.1}$$

at all points of differentiability of $H(u, t, x)$ with respect to u . Then we are going to prove it in the original form. Take an H satisfying only Assumption 1.1, take $n > 0$, and consider the mapping $T_n : w \rightarrow v$ defined for any $w \in C(\bar{\Omega}_T)$ and mapping it into a unique solution of

$$\partial_t u + \max(H[v] - v + n\chi(w/n), P[v] - K) = 0 \tag{2.2}$$

in Ω (a.e.) with terminal-boundary condition $v = g$ on $\partial' \Omega_T$, where

$$\chi(t) = (-1) \vee t \wedge 1.$$

By assumption v is well defined and $v = T_n w \in C^{1,1}(\bar{\Omega}_T) \cap W_{\infty, \text{loc}}^{1,2}(\Omega_T)$ and satisfies

$$|v|, |D_i v|, \rho |D_{ij} v|, |\partial_t v| \leq N(\bar{H} + n + K + \|g\|_{C^{1,2}(\Omega_T)}),$$

(a.e.) in Ω_T , and

$$\|v\|_{W_p^{1,2}(\Omega_T)} \leq N_p(\bar{H} + n + K + \|g\|_{W_p^{1,2}(\Omega_T)})$$

if $p > d + 1$. It follows that, for each n , T_n maps $C(\bar{\Omega}_T)$ into its compact subset.

Lemma 2.1. *For each n , the mapping T_n is continuous in $C(\bar{\Omega}_T)$.*

Proof. Let $w, w_m \in C(\bar{\Omega}_T)$, $m = 1, 2, \dots$, and assume that $\|w - w_m\|_{0, \Omega_T} \rightarrow 0$ as $m \rightarrow \infty$, where $\|\cdot\|_{0, \Omega_T}$ is the sup norm in $C(\bar{\Omega}_T)$. In light of uniqueness of solutions of (2.2) with terminal-boundary condition $v = g$, to prove the lemma, it suffices to show that, at least along a subsequence, $\|T_n w - v_m\|_{0, \Omega_T} \rightarrow 0$, where $v_m = T_n w_m$. Since $T_n C(\bar{\Omega}_T)$ is a compact set, there is a subsequence and a $v \in C(\bar{\Omega}_T)$ such that $\|v - v_m\|_{0, \Omega_T} \rightarrow 0$ and $v = g$ on $\partial' \Omega_T$. Without losing generality we may assume that the above convergence holds along the original sequence. Now we need only show that $v = T_n w$.

Observe that for $m \geq r$ we have

$$\partial_t v_m + \max(H[v_m] - v_m + n \sup_{k \geq r} \chi(w_k/n), P[v_m] - K) \geq 0$$

in Ω_T (a.e.). Since the norms $\|v_m\|_{W_{d+1}^{1,2}(\Omega_T)}$ are bounded, by Theorem 3.5.9 of [8], whose conditions are easily checked on the basis of Remark 1.2, we have (a.e.)

$$\partial_t v + \max(H[v] - v + n \sup_{k \geq r} \chi(w_k/n), P[v] - K) \geq 0.$$

By letting $r \rightarrow \infty$ we get (a.e.)

$$\partial_t v + \max(H[v] - v + n \chi(w/n), P[v] - K) \geq 0.$$

One obtains the opposite inequality starting with

$$\partial_t v_m + \max(H[v_m] - v_m + n \inf_{k \geq r} \chi(w_k/n), P[v_m] - K) \leq 0.$$

It follows that $v = T_n w$ indeed and the lemma is proved. \square

Now by Tikhonov's theorem we conclude that, for each n , there exists $v^n \in C(\bar{\Omega}_T)$ such that $v^n = T_n v^n$. By assumption $v^n \in \mathcal{C}^{1,1}(\bar{\Omega}_T) \cap W_{\infty, \text{loc}}^{1,2}(\Omega_T)$ and

$$|v^n|, |D_i v^n|, \rho |D_{ij} v^n|, |\partial_t v^n| \leq N(\bar{H} + \|v^n\|_{0, \Omega_T} + K + \|g\|_{C^{1,2}(\Omega_T)}) \quad (2.3)$$

(a.e.) in Ω_T and

$$\|v^n\|_{W_p^{1,2}(\Omega_T)} \leq N_p(\bar{H} + \|v^n\|_{0, \Omega_T} + K + \|g\|_{W_p^{1,2}(\Omega_T)}), \quad (2.4)$$

where N only depends on Ω and δ , and N_p only depends on Ω , T , δ , and p .

Lemma 2.2. *There is a constant N depending only on the diameter of Ω and δ such that*

$$\|v^n\|_{C(\Omega_T)} \leq N(\bar{H} + K + \|g\|_{C(\Omega_T)}).$$

Proof. Introduce

$$H_K^n(u, t, x) = \max(H(u, t, x) - u'_0 + n \chi(u'_0/n), P(u) - K)$$

and observe that $H_{K u'_0}^n \leq 0$ and by Hadamard's formula

$$H_K^n(u', u'', t, x) - H_K^n(0, t, x) = u''_{ij} \int_0^1 H_{K u''_{ij}}^n(\theta u', \theta u'', t, x) d\theta$$

$$+ \sum_{i \geq 1} u'_i \int_0^1 H_{Ku'_i}^n(\theta u', \theta u'', t, x) d\theta + u'_0 \int_0^1 H_{Ku'_0}^n(\theta u', \theta u'', t, x) d\theta. \quad (2.5)$$

provided that $H^n(u, t, x)$ is differentiable with respect to u at $(\theta u, x)$ for almost all $\theta \in [0, 1]$. Since this happens to be the case for almost all u , we see that, for each n , there exist \mathcal{S}_δ -valued function a and real-valued functions b_1, \dots, b_d, c , and f satisfying $|b_i| \leq \delta^{-1}$, $c \geq 0$, $|f| \leq \bar{H} + K$ such that in Ω (a.e.)

$$\partial_t v^n + a_{ij} D_{ij} v^n + b_i D_i v^n - c v^n = f.$$

Now our result follows by the parabolic Alexandrov maximum principle (see, for instance, Section 3.3 of [8]) and using the global barrier function given, for instance, in the proof of Lemma 8.8 of [10]. The lemma is proved. \square

Due to this lemma one can drop $\|v^n\|_{0,\Omega}$ on the right-hand sides of estimates (2.3) and (2.4). After that it only remains to observe that for $n \geq \|v^n\|_{0,\Omega}$, the function v^n satisfies (1.2) since $\chi(v^n/n) = v^n/n$ and Theorem 1.3 holds in its original form.

Hence, in the rest of the article we suppose that (2.1) holds at all points of differentiability of H with respect to u .

3. FURTHER REDUCTIONS OF THEOREM 1.3

1. First, we show that we may additionally assume that for any $s, t \in \mathbb{R}$, $x, y \in \mathbb{R}^d$ and $u = (u', u'')$

$$|H(u, t, x) - H(u, s, y)| \leq N(|t - s| + |x - y|)(1 + |u|), \quad (3.1)$$

where N is independent of t, s, x, y, u .

Indeed, if Theorem 1.3 is true in this particular case, take a nonnegative $\zeta \in C_0^\infty(\mathbb{R}^{d+1})$, which integrates to one, set $\zeta^n(x) = n^{d+1} \zeta(nt, nx)$, and introduce $H^n(u, t, x)$ as the convolution of $H(u, t, x)$ and ζ^n performed with respect to (t, x) . Observe that H^n satisfies (2.1) and Assumption 1.1 with the same constant δ , whereas

$$|H^n(u, t, x) - H^n(u, s, y)| \leq n(|t - s| + |x - y|) \sup_{r,z} |H(u, r, z)| \|\zeta\|_{C^1(\mathbb{R}^{d+1})}$$

and (3.1) is satisfied since

$$|H(u, r, z)| \leq |H(0, r, z)| + N(d)\delta^{-1}|u|.$$

Then assuming that the assertions of Theorem 1.3 are true under our additional assumption, we conclude that there exist solutions $v^n \in \mathcal{C}^{1,1}(\bar{\Omega}_T) \cap W_{\infty, \text{loc}}^{1,2}(\Omega_T)$ of

$$\partial_t v^n + \max(H^n[v^n], P[v^n] - K) = 0$$

in Ω_T (a.e.) with terminal-boundary condition $v^n = g$, for which estimates (1.3) and (1.4) hold with v^n in place of v with the constants N and N_p from Theorem 1.3 and with

$$\bar{H}^n = \sup_{(t,x) \in \mathbb{R}^{d+1}} |H^n(0, t, x)| \quad (\leq \bar{H})$$

in place of \bar{H} . In particular,

$$\partial_t v^m + \check{H}_K^n[v^m] \geq 0 \quad (3.2)$$

in Ω_T (a.e.) for all $m \geq n$, where

$$\check{H}_K^n(u, t, x) := \sup_{k \geq n} \max(H^k(u, t, x), P(u) - K).$$

Furthermore, being uniformly bounded and uniformly continuous, the sequence $\{v^n\}$ has a subsequence uniformly converging to a function v , for which (1.3) and (1.4), of course, hold and $v \in \mathcal{C}^{1,1}(\bar{\Omega}_T) \cap W_{\infty, \text{loc}}^{1,2}(\Omega_T)$. In light of (3.2) and the fact that the norms $\|v^n\|_{W_p^{1,2}(\Omega_T)}$ are bounded, by Theorem 3.5.9 of [8] (the applicability of which is shown by an argument similar to the one in Remark 1.4) we have

$$\partial_t v + \check{H}_K^n[v] \geq 0$$

in Ω_T (a.e.).

Then we notice that by the Lebesgue differentiation theorem for any u

$$\lim_{n \rightarrow \infty} \check{H}_K^n(u, t, x) = \max(H(u, t, x), P(u) - K) \quad (3.3)$$

for almost all (t, x) . Since $\check{H}_K^n(u, t, x)$ are Lipschitz continuous in u with a constant independent of t, x , and n , there exists a subset of Ω_T of full measure such that (3.3) holds on this subset for all u .

We conclude that in Ω_T (a.e.)

$$\partial_t v + \max(H[v], P[v] - K) \geq 0.$$

The opposite inequality is obtained by considering

$$\inf_{k \geq n} \max(H^k(u, t, x), P(u) - K).$$

2. Next, we show that one may assume that H is boundedly inhomogeneous with respect to u . Introduce

$$P_0(u) = \max_{a \in \mathcal{S}_{\delta/2}} \max_{\substack{|b_i| \leq 2\delta^{-1} \\ i=1, \dots, d}} \max_{c \in [\delta/2, 2\delta^{-1}]} (a_{ij}u''_{ij} + b_i u'_i - cu'_0),$$

where the summations are performed before the maximum is taken. It is easy to see that $P_0[u]$ is a kind of Pucci's operator:

$$\begin{aligned} P_0(u) &= -(\delta/2) \sum_{k=1}^d \lambda_k^-(u'') + 2\delta^{-1} \sum_{k=1}^d \lambda_k^+(u'') \\ &\quad + 2\delta^{-1} \sum_{k=1}^d |u'_k| - (\delta/2)(u'_0)^+ + 2\delta^{-1}(u'_0)^-, \end{aligned}$$

where $\lambda_1(u''), \dots, \lambda_d(u'')$ are the eigenvalues of u'' and $a^\pm = (1/2)(|a| \pm a)$.

Recall that the function P is introduced in the end of Section 1 and observe that

$$P(u) = \max_{\substack{\hat{\delta}/2 \leq a_k \leq 2\hat{\delta}^{-1} \\ k=1, \dots, m}} \max_{\substack{|b_i| \leq 2\hat{\delta}^{-1} \\ i=1, \dots, d}} \max_{\hat{\delta}/2 \leq c \leq 2\hat{\delta}^{-1}} \left[\sum_{i,j=1}^d \sum_{k=1}^m a_k l_{ki} l_{kj} u''_{ij} + \sum_{i=1}^d b_i u'_i - cu'_0 \right].$$

Moreover, owing to property (ii) in the end of Section 1, the collection of matrices

$$\sum_{k=1}^m a_k l_k l_k^*$$

such that $\hat{\delta} \leq a_k \leq \hat{\delta}^{-1}$, $k = 1, \dots, m$, covers $\mathcal{S}_{\delta/4}$. By combining this with the fact that $\hat{\delta} \leq \delta/2$ (actually, $\hat{\delta} \leq \delta/4$, which will be used much later) we see that

$$\begin{aligned} P(u) &\geq -(\delta/4) \sum_{k=1}^d \lambda_k^-(u'') + 4\delta^{-1} \sum_{k=1}^d \lambda_k^+(u'') \\ &\quad + 4\delta^{-1} \sum_{k=1}^d |u'_k| - (\delta/4)(u'_0)^+ + 4\delta^{-1}(u'_0)^- \\ &\geq P_0(u) + (\delta/4) \sum_{k=1}^d |\lambda_k(u'')| + (\delta/4) \sum_{k=0}^d |u'_k|. \end{aligned} \quad (3.4)$$

In particular, $P_0 \leq P$ and therefore,

$$\max(H, P - K) = \max(H_K, P - K),$$

where $H_K = \max(H, P_0 - K)$. It is easy to see that the function H_K satisfies Assumption 1.1 and (2.1) with $\delta/2$ in place of δ . It also satisfies (3.1) with the same constant N .

Furthermore, we have the following.

Lemma 3.1. *There is a constant $\kappa > 0$ depending only on δ and d such that for all $(t, x) \in \Omega_T$ and $u = (u', u'')$*

$$H \leq P_0 - \kappa \left(\sum_{i,j} |u''_{ij}| + \sum_i |u'_i| \right) + H(0, t, x), \quad (3.5)$$

$$H_K \leq P - \kappa \left(\sum_{i,j} |u''_{ij}| + \sum_i |u'_i| \right) + H^+(0, t, x). \quad (3.6)$$

Furthermore,

$$H(u, t, x) \leq N \left(\sum_{i,j} |u''_{ij}| + \sum_i |u'_i| \right) + H(0, t, x),$$

$$|H(u, t, x)| \leq N \left(\sum_{i,j} |u''_{ij}| + \sum_i |u'_i| \right) + |H(0, t, x)|,$$

where the constant N depends only on δ .

Proof. Observe that if a number $p \in (a, b)$, $a < b$, and $y \in \mathbb{R}$, then

$$yp \leq y^+b - y^-a.$$

Then from Hadamard's formula

$$\begin{aligned} H(u', u'', t, x) - H(0, 0, t, x) &= u''_{ij} \int_0^1 H_{u''_{ij}}(su', su'', t, x) ds \\ &+ \sum_{i \geq 1} u'_i \int_0^1 H_{u'_i}(su', su'', t, x) ds + u'_0 \int_0^1 H_{u'_0}(su', su'', t, x) ds \end{aligned}$$

we obtain (see our comments regarding (2.5))

$$\begin{aligned} H(u', u'', t, x) - H(0, 0, t, x) &\leq \delta^{-1} \sum_k \lambda_k^+(u'') - \delta \sum_k \lambda_k^-(u'') \\ &+ \delta^{-1} \sum_{i \geq 1} |u'_i| - \delta(u'_0)^+ + \delta^{-1}(u'_0)^- = P_0(u', u'') \\ &- \delta^{-1} \sum_k \lambda_k^+(u'') - (\delta/2) \sum_k \lambda_k^-(u'') - \delta^{-1} \sum_{i \geq 1} |u'_i| - \delta^{-1}(u'_0)^- - (\delta/2)(u'_0)^+ \end{aligned}$$

and (3.5) follows since

$$\begin{aligned} \left[\sum_k (\lambda_k^+(u'') + \lambda_k^-(u'')) \right]^2 &= \left(\sum_k |\lambda_k(u'')| \right)^2 \\ &\geq \sum_k |\lambda_k(u'')|^2 = \sum_{i,j} |u''_{ij}|^2 \geq d^{-2} \left(\sum_{i,j} |u''_{ij}| \right)^2. \end{aligned}$$

Estimate (3.6) follows from (3.5) and (3.4). Finally, the second assertion of the lemma follows directly from the above Hadamard's formula. The lemma is proved. \square

In addition, H_K is boundedly inhomogeneous with respect to u in the sense that at all points of differentiability of $H_K(u, t, x)$ with respect to u

$$|H_K(u, t, x) - H_{Ku''_{ij}}(u, t, x)u''_{ij} - H_{Ku'_r}(u, t, x)u'_r| \leq N(|H_K(0, t, x)| + K), \quad (3.7)$$

where N depends only on δ and d .

Indeed, if

$$\kappa \left(\sum_{i,j} |u''_{ij}| + \sum_i |u'_i| \right) \geq H^+(0, t, x) + K, \quad (3.8)$$

then by Lemma 3.1

$$H(u, x) \leq P_0(u) - \kappa \left(\sum_{i,j} |u''_{ij}| + \sum_i |u'_i| \right) + H^+(0, t, x) \leq P_0(u) - K,$$

so that $H_K(u, t, x) = P_0(u) - K$ and the left-hand side of (3.7) is just K owing to the fact that P_0 is positive homogeneous of degree one. On the

other hand, if the opposite inequality holds in (3.8), then again in light of Lemma 3.1 the left-hand side of (3.7) is dominated by

$$N\left(\sum_{i,j}|u''_{ij}| + \sum_i |u'_i|\right) + |H_K(0, t, x)| \leq N(|H_K(0, t, x)| + H^+(0, t, x) + K),$$

where

$$H(0, t, x) \leq \max(H(0, t, x), -K) = H_K(0, t, x),$$

$$H^+(0, t, x) \leq |H_K(0, t, x)|.$$

Furthermore, as we have noticed above H_K satisfies Assumption 1.1 and (2.1) (with $\delta/2$ in place of δ) and as is easy to see $|H_K[0]| \leq |H[0]| + K$, which shows that in the rest of the article we may (and will) assume that not only Assumption 1.1 and (2.1) are satisfied with $\delta/2$ in place of δ and (3.1) holds with a constant N , but also at all points of differentiability of H with respect to u

$$|H(u, t, x) - H_{u''_{ij}}(u, t, x)u''_{ij} - H_{u'_r}(u, t, x)u'_r| \leq N_0, \quad (3.9)$$

where N_0 is a constant and

$$H \leq P - \kappa\left(\sum_{i,j}|u''_{ij}| + \sum_i |u'_i|\right) + |H(0, \cdot, \cdot)|, \quad (3.10)$$

where κ is the constant from Lemma 3.1. By the way we keep track of the value of δ in Assumption 1.1 and (2.1) because $P(u)$ is already fixed and defined by d and δ .

As a result of the above arguments we see that to prove Theorem 1.3 it suffices to prove the following.

Theorem 3.2. *Suppose that Assumption 1.1 is satisfied with $\delta/2$ in place of δ . Also assume that (3.10) holds. Finally, assume that estimate (3.1) holds for any $t, s \in \mathbb{R}$, $x, y \in \mathbb{R}^d$, and $u = (u', u'')$ with a constant N and (2.1) and (3.9) hold at all points of differentiability of $H(u, t, x)$ with respect to u . Then the assertions of Theorem 1.3 hold true with P introduced in the end of Section 1.*

Remark 3.3. One may wonder why we need (3.9) with a constant which does not enter the assertions of Theorem 3.2 in any way. The only reason to reduce general H to boundedly inhomogeneous ones is that for those we can rewrite $H[v]$ in such a way that only pure second-order derivatives of $v(t, x)$ with respect to x enter. Then the whole operator $\max(H[v], P[v] - K)$ also has this form.

Another possible question is: Why don't we start with $\max(H, P - K)$, which is already boundedly inhomogeneous by the above? The point is that our way to transform boundedly inhomogeneous operators does not preserve the particular structure of $\max(H, P - K)$.

4. AN AUXILIARY EQUATION

Some notation in this section are different from the previous ones. Fix an $h \in (0, 1]$ and for $\xi \in \mathbb{R}^d$ and any function ϕ on \mathbb{R}^d introduce

$$T_\xi \phi(x) = \phi(x + h\xi), \quad \delta_\xi = h^{-1}(T_\xi - 1), \quad \Delta_\xi = h^{-2}(T_\xi - 2 + T_{-\xi}).$$

Notice that h enters the definition of T_ξ and δ_ξ and Δ_ξ are usual approximations for the first and second-order derivatives along ξ .

Let $m \geq 1$ be an integer and let $\ell_{-m}, \dots, \ell_{-1}, \ell_1, \dots, \ell_m$ be some fixed vectors in \mathbb{R}^d such that

$$\ell_{-k} = -\ell_k.$$

Next denote $\Lambda = \{\ell_k : k = \pm 1, \dots, \pm m\}$,

$$\Lambda_1 = \Lambda, \quad \Lambda_{n+1} = \Lambda_n + \Lambda, \quad n \geq 1, \quad \Lambda_\infty = \bigcup_n \Lambda_n \quad \Lambda_\infty^h = h\Lambda_\infty.$$

Let $m' \geq 0$ be an integer $\leq m$ and let $A = \{\alpha = (a, b, c)\}$ be a closed bounded set in $\mathbb{R}^{2m} \times \mathbb{R}^{m'} \times \mathbb{R}$, so that

$$a = (a_{-m}, a_{-m+1}, \dots, a_{-1}, a_1, \dots, a_m) \in \mathbb{R}^{2m},$$

$$b = (b_1, \dots, b_{m'}) \in \mathbb{R}^{m'},$$

and $c \in \mathbb{R}$. Also let $f(\alpha, t, x)$ be a real-valued function defined for $\alpha \in A$, $t \in \mathbb{R}$, and $x \in \mathbb{R}^d$.

Fix an $r \in \{1, \dots, m\}$ and for $k = \pm 1, \dots, \pm m$ set

$$\delta_{h,k} = \delta_k = \delta_{\ell_k}, \quad \Delta_{h,k} = \Delta_k = \Delta_{\ell_k}.$$

Assumption 4.1. There are constants $\delta > 0$ and $K_1, K_2 \in [0, \infty)$ such that

(i) For any $(a, b, c) \in A$ and all k we have

$$a_k = a_{-k}, \quad \delta \leq a_k \leq \delta^{-1}, \quad |b_k| \leq \delta^{-1}, \quad hb_k^- \leq a_k, \quad c \geq 0;$$

(ii) The function f is continuous in α for any (t, x) and $|\delta_r f| \leq K_1$, $\Delta_r f \geq -K_2$ on \mathbb{R}^d .

For $u = (u', u'')$ with

$$u' = (u'_0, u'_1, \dots, u'_{m'}), \quad u'' = (u''_{-m}, \dots, u''_{-1}, u''_1, \dots, u''_m),$$

introduce

$$\mathcal{P}(u, t, x) = \max_{\alpha=(a,b,c) \in A} \left(\sum_{|k|=1}^m a_k u''_k + \sum_{k=1}^{m'} b_k u'_k - cu'_0 + f(\alpha, t, x) \right).$$

For any function u on \mathbb{R}^{d+1} define

$$\mathcal{P}[u](t, x) = \mathcal{P}(u(t, x), \delta u(t, x), \delta^2 u(t, x), t, x),$$

where

$$\begin{aligned} \delta u &= (\delta_1 u, \dots, \delta_{m'} u), \\ \delta^2 u &= (\Delta_{-m} u, \dots, \Delta_{-1} u, \Delta_1 u, \dots, \Delta_m u). \end{aligned}$$

In connection with this notation a natural question arises as to why use ℓ_k along with $\ell_{-k} = -\ell_k$ since $\Delta_k = \Delta_{-k}$ and

$$a_k \Delta_k = 2 \sum_{k \geq 1} a_k \Delta_k$$

owing to the assumption that $a_k = a_{-k}$. This is done for the sake of convenience of computations. For instance,

$$\Delta_k(uv) = u\Delta_k v + v\Delta_k u + (\delta_k u)(\delta_k v) + (\delta_{-k} u)(\delta_{-k} v)$$

(no summation in k). At the same time

$$a_k \Delta_k(uv) = ua_k \Delta_k v + va_k \Delta_k u + 2a_k (\delta_k u)(\delta_k v)$$

as if we were dealing with usual partial derivatives.

Let Q^o be a bounded subset of $\mathbb{R} \times \Lambda_\infty^h$, which is open in the relative topology of $\mathbb{R} \times \Lambda_\infty^h$ and is such that its projection on Λ_∞^h is a finite set. Introduce \hat{Q}^o as the set of points $(t, x_0) \in \mathbb{R} \times \Lambda_\infty^h$ for each of which there exists a sequence $t_n \uparrow t_0$ such that $(t_n, x_0) \in Q^o$. Observe that $Q^o \subset \hat{Q}^o$. Also define

$$Q = \hat{Q}^o \cup \{(t, x + h\Lambda) : (t, x) \in Q^o\}.$$

For $x \in \Lambda_\infty^h$ we denote by $Q_{|x}^o$ the x -section of Q^o : $\{t : (t, x) \in Q^o\}$.

In the future we will need the following.

Lemma 4.2. *Let $(a, b, c)(t, x)$ be a bounded $\mathbb{R}^{2m} \times \mathbb{R}^{m'} \times \mathbb{R}$ -valued (say A -valued) function on \mathbb{R}^{d+1} satisfying $a_k \geq 0$, $hb_k^- \leq a_k$, and $c \geq 0$, and let $v(t, x)$ be a bounded function in Q which is absolutely continuous with respect to t on each open interval belonging to $Q_{|x}^o$ and for any $x \in \Lambda_\infty^h$ satisfies*

$$\partial_t v + Lv := \partial_t v + \sum_{|k|=1}^m a_k \Delta_k v + \sum_{k=1}^{m'} b_k \delta_k v - cv = -\eta$$

(a.e.) on each $Q_{|x}^o$, where $\eta = \eta(t, x)$ is a bounded function. Redefine v if necessary for $(t, x) \in \hat{Q}^o \setminus Q^o$ so that

$$v(t, x) = \overline{\lim_{s \uparrow t, (s, x) \in Q^o}} v(s, x).$$

Finally, let T be the width of Q^o in the t -direction. Then in Q^o we have

$$v \leq T \sup_{Q^o} \eta_+ + \sup_{Q \setminus Q^o} v_+.$$

Proof. Without losing generality we assume that $Q^o \in (0, T) \times \Lambda_\infty^h$. Then by considering

$$v(t, x) - (T - t)[2\varepsilon + \sup_{Q^o} \eta_+],$$

where $\varepsilon > 0$, and then sending $\varepsilon \downarrow 0$, we reduce the general case to the one with $\eta \leq -2\varepsilon$. Finally, we make one more harmless assumption that

$$\sup_Q v > 0.$$

After that take a sequence $(t_n, x_n) \in Q$ such that

$$v(t_n, x_n) \rightarrow \bar{v} := \sup_Q v > 0.$$

If infinitely many points $(t_n, x_n) \notin Q^o$, then we have nothing to prove.

In the opposite case we may assume that $x_n = x_0$, $(t_n, x_n) \in Q^o$ for all n , and the sequence t_n converges, say to t_0 . Denote by I_n the connected component (open interval) of $Q|_{x_0}$ containing t_n . By using subsequences if needed and taking into account the continuity of v in Q^o we come to three possibilities: either $(t_0, x_0) \in \hat{Q}^o \setminus Q^o$ and we have nothing to prove, or $(t_0, x_0) \in Q^o$, or else $t_n \downarrow t_0$. Note that the second case can be reduced to the third one by redefining the t_n 's.

If the third possibility realizes, we claim that

$$\lim_{n \rightarrow \infty} |I_n| = 0, \quad (4.1)$$

where $|I_n|$ is the length of I_n .

Indeed if (4.1) fails, then for all large n the intervals I_n coincide. Also in that case there is an open interval $I \in \mathbb{R}$ such that

$$I \times \{x_0\} \subset Q^o, \quad I \times \{x_0 + h\Lambda\} \subset Q.$$

Furthermore, $\partial_t v(t, x_0)$ is bounded on I , so that the limit of $v(t, x_0)$ as $t \downarrow t_0$ exists and

$$\lim_{t \downarrow t_0} v(t, x_0) = \lim_{n \rightarrow \infty} v(t_n, x_0) = \bar{v}$$

In addition, $\bar{v} \geq v(t, x_0)$ for $t > t_0$ and, since

$$\partial_t v(t, x_0) = -Lv(t, x_0) - \eta(t, x_0)$$

for almost all $t \in I$, there exists a sequence of points $s_n \in I$ such that $s_n \downarrow t_0$ and

$$Lv(s_n, x_0) + \eta(s_n, x_0) \geq -\varepsilon$$

implying that (recall that $\eta \leq -2\varepsilon$)

$$Lv(s_n, x_0) \geq \varepsilon. \quad (4.2)$$

Next, consider the functions $v_n(x) = v(s_n, x)$, $x \neq x_0$, $v_n(x_0) = \bar{v}$, for which $v_n(x_0) \geq v_n(x)$ for all $x \in x_0 + h\Lambda$. On the one hand, by the maximum principle we have $Lv_n(s_n, x_0) \leq 0$ and, on the other hand

$$Lv_n(s_n, x_0) = Lv(s_n, x_0) + \xi_n,$$

where

$$\begin{aligned} \xi_n &= 2h^{-2} \sum_{|k|=1}^m a_k(s_n, x_0)[v(s_n, x_0) - \bar{v}] \\ &\quad + h^{-1} \sum_{k=1}^{m'} b_k(s_n, x_0)[v(s_n, x_0) - \bar{v}] + c(s_n, x_0)[v(s_n, x_0) - \bar{v}] \end{aligned}$$

and $\xi_n \rightarrow 0$ as $n \rightarrow \infty$. This leads to a contradiction with (4.2) and proves (4.1).

It follows that for infinitely many n , as n increases, the value of v at (t_n, x_0) will become closer and closer to its value at the right end points of I_n 's since the time derivative of v is bounded and this proves the lemma. \square

Next, take a function $\eta \in C^\infty(\mathbb{R}^d)$ with bounded derivatives, such that $|\eta| \leq 1$ and set $\zeta = \eta^2$,

$$|\eta'(x)| = |\eta'(x)|_h = \sup_k |\delta_k \eta(x)|, \quad |\eta''(x)| = |\eta''(x)|_h = \sup_k |\Delta_k \eta(x)|,$$

$$\|\eta'\| = \|\eta'\|_h = \sup_{\Lambda_\infty^h} |\eta'|_h, \quad \|\eta''\| = \|\eta''\|_h = \sup_{\Lambda_\infty^h} |\eta''|_h,$$

Finally, let u be a function on \mathbb{R}^{d+1} which is continuously differentiable with respect to t and satisfies

$$\partial_t u + \mathcal{P}[u] = 0 \quad \text{in } Q^o \quad (4.3)$$

and

$$\partial_t u + \mathcal{P}[u] \leq 0 \quad \text{on } Q \setminus Q^o. \quad (4.4)$$

Theorem 4.3. *There exist constants $N = N(m, \delta) \geq 1$ and $N^* = N^*(m, \delta)$ such that for any constant ν satisfying*

$$\nu \geq N^* \|\eta'\| + N(\|\eta''\| + \|\eta'\|^2),$$

we have in Q^o that (recall that $a^\pm = (1/2)(|a| \pm a)$)

$$\zeta^2 [(\Delta_r u)^-]^2 \leq \sup_{Q \setminus Q^o} \zeta^2 [(\Delta_r u)^-]^2 + (N\nu + N^*) \bar{W}_r + N\nu^{-2} K_2^2 + \nu^{-1} K_1^2, \quad (4.5)$$

where

$$\bar{W}_r = \sup_Q (|\delta_r u|^2 + |\delta_{-r} u|^2).$$

Furthermore, $N^ = 0$ if $b \equiv 0$.*

In the remaining part of this section no summation with respect to r is performed. The number r is fixed at the beginning of the section. For simplicity of notation set

$$u_{rr} = \Delta_r u, \quad u_r = \delta_r u, \quad u_{kr} = -\delta_{-k} \delta_r u.$$

Notice that in the above line the last notation when $k = r$ is consistent with the first one.

Define

$$u_{rr}^- = (u_{rr})^-.$$

and for a constant $\nu \geq 0$ introduce an operator (recall that r is fixed)

$$L_\nu \phi = \zeta^2 u_{rr}^- \Delta_r \phi - \nu \zeta u_r \delta_r \phi.$$

Observe that

$$L_\nu u = -\zeta^2 (u_{rr}^-)^2 - \nu \zeta u_r^2 =: -V_\nu. \quad (4.6)$$

In the following lemma the fact that u is a solution of (4.3) is not used.

Lemma 4.4. *There exists $N = N(m, \delta) \geq 1$ and $N^* = N^*(m, \delta)$ such that if*

$$\nu \geq N^* \|\eta'\| + N(\|\eta''\| + \|\eta'\|^2) \quad (4.7)$$

*and $N^*h \leq 1$, then on Q^o for any $\alpha = (a, b, c) \in A$ we have*

$$\begin{aligned} 2L_\nu[\partial_t + a_k\Delta_k + b_k\delta_k]u &\geq -[\partial_t + a_k\Delta_k + b_k\delta_k]V_\nu \\ &\quad - (N\nu^2 + N^*\nu)\bar{W}_r + (\nu/2)\zeta a_k u_{kr}^2. \end{aligned} \quad (4.8)$$

Furthermore, $N^ = 0$ if $b \equiv 0$.*

Up to an obvious formula $2L_\nu\partial_t u = -\partial_t V_\nu$ this lemma is identical to Lemma 5.3 of [13].

Proof of Theorem 4.3. Denote by N_0 and N_0^* the constants N and N^* in Lemma 4.4 and take and fix a ν satisfying (4.7) (with N_0 and N_0^* in place of N and N^*).

Notice that in Q^o

$$|u_{rr}| = h^{-1}|u_r + u_{-r}| \leq 2h^{-1}\bar{W}_r^{1/2},$$

which shows that (4.5) holds if $h \geq \nu^{-1/2}$ or if $N_0^*h \geq 1$. Therefore below we assume that

$$h \leq \nu^{-1/2}, \quad N_0^*h \leq 1. \quad (4.9)$$

Let $(t_0, x_0) \in \bar{Q}$ be a point such that

$$V_\nu(t_0, x_0) = \sup_Q V_\nu.$$

If $(t_0, x_0) \notin \bar{Q}^o$, then, as is easy to see, this point can be approximated by points lying in $Q \setminus Q^o$, in which case

$$\sup_Q V_\nu = \sup_{Q \setminus Q^o} V_\nu$$

and (4.5) follows. Therefore, in the rest of the proof we may assume that

$$(t_0, x_0) \in \bar{Q}^o.$$

We may also assume that

$$\zeta(x_0)u_{rr}^-(t_0, x_0) > \nu h u_r(t_0, x_0). \quad (4.10)$$

Indeed, if the opposite inequality holds, then in light of (4.9) in Q^o

$$\zeta^2[u_{rr}^-]^2 \leq V_\nu(t_0, x_0) \leq \nu^2 h^2 u_r^2(t_0, x_0) + \nu \bar{W}_r \leq 2\nu \bar{W}_r.$$

Next, consider two cases: 1) $\partial_t V_\nu(t_0, x_0) > 0$, 2) $\partial_t V_\nu(t_0, x_0) \leq 0$. In the first case $(t_0, x_0) \notin Q^o$ and there is no sequence $t_n \downarrow t_0$ such that $(t_n, x_0) \in Q^o$. Hence $(t_0, x_0) \in \hat{Q}^o \setminus Q^o$ and in Q^o

$$\zeta^2[u_{rr}^-]^2 \leq V_\nu(t_0, x_0) \leq \sup_{\hat{Q}^o \setminus Q^o} \zeta^2[(\Delta_r u)^-]^2 + \nu \bar{W}_r,$$

so that (4.5) holds.

In the remaining case

$$\partial_t V_\nu(t_0, x_0) \leq 0. \quad (4.11)$$

To extract some consequences of (4.11), first we notice that if a function $\phi(t_0, x)$ is such that $\phi(t_0, x) \leq \phi(t_0, x_0)$ for $x \in x_0 + h\Lambda$, then owing to (4.10) at (t_0, x_0) we have

$$\begin{aligned} h^2 L_\nu \phi(t_0, x_0) &= \zeta[\phi(x_0 + h\ell_r)(\zeta u_{rr}^- - \nu h u_r) + \phi(x_0 - h\ell_r)\zeta u_{rr}^-] \\ &\quad - \zeta[2\zeta u_{rr}^- - \nu h u_r]\phi \leq \zeta[(\zeta u_{rr}^- - \nu h u_r)\phi + \zeta u_{rr}^- \phi] - \zeta[2\zeta u_{rr}^- - \nu h u_r]\phi, \end{aligned}$$

where the last expression is zero. Thus

$$L_\nu \phi(t_0, x_0) \leq 0,$$

which in the terminology from [11] means that L_ν respects the maximum principle.

Furthermore, we can find an $\bar{\alpha} = (\bar{a}, \bar{b}, \bar{c}) \in A$ such that

$$\begin{aligned} \partial_t u(t_0, x_0) + \bar{a}_k \Delta_k u(t_0, x_0) + \bar{b}_k \delta_k u(t_0, x_0) - \bar{c} u(t_0, x_0) + f(\bar{\alpha}, t_0, x_0) \\ = \partial_t u(t_0, x_0) + \mathcal{P}[u](t_0, x_0) = 0. \end{aligned}$$

Since $\partial_t u + \mathcal{P}[u] \leq 0$ in Q , we have that

$$\begin{aligned} \phi(t_0, x) := \partial_t u(t_0, x) + \bar{a}_k \Delta_k u(t_0, x) + \bar{b}_k \delta_k u(t_0, x) - \bar{c} u(t_0, x) + f(\bar{\alpha}, t_0, x) \leq 0 \\ \text{for } x \in x_0 + h\Lambda. \text{ Hence, } 0 \geq 2L_\nu \phi(t_0, x_0), \text{ which owing to (4.6), (4.8), and} \\ \text{(4.11) yields} \end{aligned}$$

$$\begin{aligned} 0 &\leq [\partial_t + \bar{a}_k \Delta_k + \bar{b}_k \delta_k - 2\bar{c}]V_\nu(t_0, x_0) - (\nu/2)\zeta \bar{a}_k u_{kr}^2(t_0, x_0) \\ &\quad + (N\nu^2 + N^*\nu)\bar{W}_r - 2L_\nu f(\bar{\alpha}, \cdot)(t_0, x_0) \\ &\leq [\bar{a}_k \Delta_k + \bar{b}_k \delta_k - 2\bar{c}]V_\nu(t_0, x_0) - (\nu/2)\zeta \bar{a}_k u_{kr}^2(t_0, x_0) \\ &\quad + (N\nu^2 + N^*\nu)\bar{W}_r - 2L_\nu f(\bar{\alpha}, \cdot)(t_0, x_0). \end{aligned}$$

Here the last term is dominated by

$$\begin{aligned} K_2 \zeta^2 u_{rr}^-(t_0, x_0) + \nu |u_r(t_0, x_0)| K_1 \\ \leq N\nu^{-1} K_2^2 + (\nu/4)\zeta \bar{a}_k u_{kr}^2(t_0, x_0) + K_1^2 + \nu^2 \bar{W}_r. \end{aligned}$$

Furthermore, since $V_\nu(t_0, x) \geq 0$ attains its maximum at (t_0, x_0) ,

$$[\bar{a}_k \Delta_k + \bar{b}_k \delta_k - 2\bar{c}]V_\nu(t_0, x_0) \leq 0.$$

We now conclude that

$$(\nu/4)\zeta \bar{a}_k u_{kr}^2(t_0, x_0) \leq (N\nu^2 + N^*\nu)\bar{W}_r + N\nu^{-1} K_2^2 + K_1^2,$$

which implies that in Q^o

$$\begin{aligned} \zeta^2 (u_{rr}^-)^2 &\leq V_\nu(t_0, x_0) \leq N\zeta \bar{a}_k u_{kr}^2(t_0, x_0) + \nu \bar{W}_r \\ &\leq (N\nu + N^*)\bar{W}_r + N\nu^{-2} K_2^2 + \nu^{-1} K_1^2. \end{aligned}$$

Thus, estimate (4.5) holds on Q^o in all cases and this proves the theorem. \square

5. A MODEL CUT-OFF EQUATION

We will work in the setting of Section 4. However now $h > 0$ is not fixed. Take a function $\mathcal{H}(u, t, x)$, where $(t, x) \in \mathbb{R}^{d+1}$, $u = (u', u'') \in \mathbb{R}^{1+m'+2m}$.

Assumption 5.1. (i) The function \mathcal{H} is Lipschitz continuous in u for every (t, x) , and at all points of differentiability of \mathcal{H} with respect to u we have

$$\delta \leq \mathcal{H}_{u_k''} \leq \delta^{-1}, \quad k = \pm 1, \dots, \pm m, \quad \delta \leq -\mathcal{H}_{u_0'} \leq \delta^{-1},$$

$$|\mathcal{H}_{u_k'}| \leq \delta^{-1}, \quad k = 1, \dots, m';$$

(ii) The number $\bar{\mathcal{H}} = \sup_{t,x} |\mathcal{H}(0, 0, t, x)|$ is finite;

(iii) The function \mathcal{H} is locally Lipschitz continuous in (t, x) for every u and there exists a constant N' such that at all points of differentiability of \mathcal{H} with respect to (t, x) we have

$$|\partial_t \mathcal{H}(u, t, x)| + |\mathcal{H}_{x_i}(u, t, x)| \leq N'(1 + |u|), \quad \forall i;$$

(iv) We have $\text{Span}(\ell_1, \dots, \ell_m) = \mathbb{R}^d$.

Define

$$\begin{aligned} \mathcal{P}(u', u'', t, x) &= \mathcal{P}(u', u'') = 2\delta^{-1} \sum_k (u_k'')^+ - (\delta/2) \sum_k (u_k'')^- \\ &\quad + 2\delta^{-1} \sum_{k \geq 1} |u_k'| - (\delta/2)(u_0')^+ + 2\delta^{-1}(u_0')^- \\ &= \max_{\substack{\delta/2 \leq a_k \leq 2/\delta \\ |k|=1, \dots, m}} \max_{\substack{|b_k| \leq 2/\delta \\ |k|=1, \dots, m'}} \max_{\delta/2 \leq c \leq 2/\delta} \left[\sum_{|i|=1}^m a_i u_i'' + \sum_{i=1}^{m'} b_i u_i' - c u_0' \right]. \end{aligned}$$

For functions $v(t, x)$ introduce

$$H[v](t, x) = \mathcal{H}(v(t, x), \partial v(t, x), \partial^2 v(t, x), t, x)$$

whenever and wherever it makes sense, where

$$\partial v = (v_{(\ell_1)}, \dots, v_{(\ell_{m'})}),$$

$$\partial^2 v = (v_{(\ell_{-m})(\ell_{-m})}, \dots, v_{(\ell_{-1})(\ell_{-1})}, v_{(\ell_1)(\ell_1)}, \dots, v_{(\ell_m)(\ell_m)}),$$

and $v_{(\ell)} = \ell_i v_{x_i}$, $v_{(\ell)}(\ell) = \ell_i \ell_j v_{x_i x_j}$. Similarly,

$$P[u](t, x) = \mathcal{P}(u(t, x), \partial u(t, x), \partial^2 u(t, x)).$$

Let $T \in (0, \infty)$, Ω be a bounded C^2 domain in \mathbb{R}^d , $g \in C^{1,2}(\bar{\Omega}_T)$, and let $K \geq 0$ be a finite number.

Theorem 5.2. *In addition to Assumption 5.1 suppose that $\pm e_i, \pm(e_i + e_j), e_i - e_j \in \Lambda$, $i, j = 1, \dots, d$, where e_1, \dots, e_d is the standard orthonormal basis in \mathbb{R}^d and assume that all vectors in Λ have rational coordinates. Then there exists a unique $v \in C^{1,1}(\bar{\Omega}_T) \cap W_{\infty, loc}^{1,2}(\Omega_T)$ such that $v = g$ on $\partial\Omega_T$ and*

$$\partial_t v + H_K[v] = 0 \tag{5.1}$$

(a.e.) in Ω_T , where

$$H_K[v] = \max(H[v], P[v] - K).$$

Furthermore,

$$|v|, |D_i v|, \rho |D_{ij} v|, |\partial_t v| \leq N(\bar{\mathcal{H}} + K + \|g\|_{C^{1,2}(\Omega_T)}) \quad (5.2)$$

in Ω_T (a.e.) for all i, j , where N is a constant depending only on Ω , $\{\ell_1, \dots, \ell_m\}$, d , and δ (but not on N').

To prove the theorem, we are going to use finite-difference approximations of the operators $H[v]$ and $P[v]$.

For $h > 0$ introduce

$$P_h[v](t, x) = \mathcal{P}(v(t, x), \delta_h v(t, x), \delta_h^2 v(t, x)),$$

where

$$\begin{aligned} \delta_h v &= (\delta_{h,1} v, \dots, \delta_{h,m'} v), \\ \delta_h^2 v &= (\Delta_{h,-m} v, \dots, \Delta_{h,-1} v, \Delta_{h,1} v, \dots, \Delta_{h,m} v). \end{aligned}$$

Similarly we introduce

$$H_h[v](t, x) = H(v(t, x), \delta_h v(t, x), \delta_h^2 v(t, x))$$

and $H_{K,h}[v] = \max(H_h[v], P_h[v] - K)$.

Here is Lemma 6.2 of [13]. Its proof is similar to that of Lemma 3.1.

Lemma 5.3. *Under Assumptions 5.1 (i) and (ii),*

$$\mathcal{H} \leq \mathcal{P} - (\delta/4) \left(\sum_k |u_k''| + \sum_k |u_k'| \right) + \bar{\mathcal{H}}.$$

Introduce B as the smallest closed ball containing Λ and set

$$\Omega^h = \{x \in \Omega : x + hB \subset \Omega\} = \{x : \rho(x) \geq \lambda h\},$$

where λ is the radius of B .

Consider the equation

$$\partial_t v + H_{K,h}[v] = 0 \quad \text{in} \quad [0, T] \times \Omega^h \quad (5.3)$$

with terminal-boundary condition

$$v = g \quad \text{on} \quad \left(\{T\} \times \Omega^h \right) \cup \left((0, T) \times (\Omega \setminus \Omega^h) \right). \quad (5.4)$$

In view of Picard's method of successive approximations, for any $h > 0$ there exists a unique bounded solution $v = v_h$ of (5.3)–(5.4). Furthermore, $\partial_t v_h$ is bounded and continuous. By the way, we do not include K in the notation v_h since K is a fixed number.

Below by h_0 and N with occasional indices we denote various (finite) constants depending only on Ω , $\{\ell_1, \dots, \ell_m\}$, d , and δ .

In the following lemma the additional assumption of Theorem 5.2 concerning the e_i 's is not used.

Lemma 5.4. *Suppose that all vectors in Λ have rational coordinates and that Assumptions 5.1 (i), (ii), and (iv) are satisfied. Then there are constants $h_0 > 0$ and N such that for all $h \in (0, h_0]$ and $|r| = 1, \dots, m$*

$$|v_h - g| \leq N(\bar{\mathcal{H}} + K + \|g\|_{C^{1,2}(\Omega_T)})\rho, \quad (5.5)$$

$$|\delta_{h,r}v_h| \leq N(\bar{\mathcal{H}} + K + \|g\|_{C^{1,2}(\Omega_T)}), \quad (5.6)$$

$$|\partial_t v_h| \leq N(\bar{\mathcal{H}} + K + \|g\|_{C^{1,2}(\Omega_T)}) \quad (5.7)$$

on Ω_T .

Proof. Introduce

$$\mathcal{H}_K = \max(\mathcal{H}, \mathcal{P} - K).$$

As is easy to see, \mathcal{H}_K satisfies Assumption 5.1 with $\delta/2$ in place of δ . Therefore, by Hadamard's formula (cf. our comments about (2.5)) there exist functions $a_k, b_k, k = \pm 1, \dots, \pm m$, and c such that

$$\delta/2 \leq a_k \leq 2\delta^{-1}, \quad |b_k| \leq 2\delta^{-1}, \quad \delta/2 \leq c \leq 2\delta^{-1} \quad (5.8)$$

and in $(0, T) \times \Omega^h$ we have

$$\begin{aligned} -\mathcal{H}_K[0] &= \partial_t v_h + H_{K,h}[v_h] - \mathcal{H}_K[0] = \partial_t v_h + a_k \Delta_{h,k} v_h + b_k \delta_{h,k} v_h - c v_h \\ &= \partial_t (v_h - g) + a_k \Delta_{h,k} (v_h - g) + b_k \delta_{h,k} (v_h - g) - c (v_h - g) + f, \end{aligned}$$

where

$$f = \partial_t g + a_k \Delta_{h,k} g + b_k \delta_{h,k} g - c g.$$

After that (5.5) is proved by using the barrier function Φ from Lemma 8.8 of [10] and the comparison principle (see, for instance, Section 5 of [5]). In particular, (5.5) implies that

$$|v_h - g| \leq N_1(\bar{\mathcal{H}} + K + \|g\|_{C^{1,2}(\Omega_T)})h \quad \text{on } (0, T) \times (\Omega \setminus \Omega^{3h}). \quad (5.9)$$

Clearly, the remaining assertion of the lemma would follow if we can prove that (5.6) and (5.7) hold on $\Omega_T \cap [(0, T) \times (y + \Lambda_\infty^h)]$ for any $y \in \mathbb{R}^d$ with a constant N independent of h and y . Without losing generality we concentrate on $y = 0$ and observe that the number of points in $\Omega^{2h} \cap \Lambda_\infty^h$ is finite since the ℓ_k 's have rational coordinates.

To prove (5.6), fix an r and define

$$Q^o = \{(t, x) \in (0, T) \times [\Omega^{2h} \cap \Lambda_\infty^h] : (\delta/4)|\delta_{h,r}v_h| > \bar{\mathcal{H}} + K\}.$$

If $Q^o = \emptyset$, then $(\delta/4)|\delta_{h,r}v_h| \leq \bar{\mathcal{H}} + K$ in $(0, T) \times [\Omega^{2h} \cap \Lambda_\infty^h]$, and by virtue of (5.9),

$$|\delta_{h,r}(v_h - g)| \leq 2N_1(\bar{\mathcal{H}} + K + \|g\|_{C^{1,2}(\Omega_T)})$$

in $(0, T) \times (\Omega \setminus \Omega^{2h})$. In that case (5.6) obviously holds.

Therefore, we assume that $Q^o \neq \emptyset$ and owing to Lemma 5.3 conclude that

$$\partial_t v_h + P_h[v_h] = K \quad (5.10)$$

in Q^o . Furthermore, (5.3) implies that

$$\partial_t v_h + P_h[v_h] \leq K \quad (5.11)$$

in $(0, T) \times \Omega^h$. Now use again the mean value theorem to conclude that

$$\delta_{h,r} P_h[v_h] = a_k \Delta_{h,k}(\delta_{h,r} v_h) + b_k \delta_{h,k}(\delta_{h,r} v_h) - c(\delta_{h,r} v_h)$$

for some functions a_k , b_k , and c satisfying (5.8). In addition,

$$\delta_{h,r}(\partial_t v_h + P_h[v_h]) \leq 0$$

in Q^o owing to (5.10) and (5.11), that is in Q^o

$$\partial_t \delta_{h,r} v_h + a_k \Delta_{h,k}(\delta_{h,r} v_h) + b_k \delta_{h,k}(\delta_{h,r} v_h) - c(\delta_{h,r} v_h) \leq 0.$$

For small enough h_0 the operator $\partial_t + a_k \Delta_{h,k} + b_k \delta_{h,k} - c$ with $h \in (0, h_0]$ respects the maximum principle and therefore by Lemma 4.2

$$\sup_{Q^o} (\delta_{h,r} v_h)_+ \leq \sup_{(0,T] \times [\Omega \cap \Lambda_\infty^h] \setminus Q^o} (\delta_{h,r} v_h)_+. \quad (5.12)$$

While estimating the right-hand side of (5.12), notice that if $(t, x) \in (0, T] \times [\Omega \cap \Lambda_\infty^h] \setminus Q^o$, then one of the following happens:

- (i) $t = T$,
- (ii) $t < T$ and $(t, x) \notin (0, T) \times \Omega^{2h}$,
- (iii) $t < T$ and $(t, x) \in (0, T) \times \Omega^{2h}$ and $(\delta/2)|\delta_{h,r} v_h| \leq \bar{\mathcal{H}} + K$.

In case (i) we have $v_h = g$, in case (ii) we may certainly use (5.5), and in case (iii) the estimate we need is just given.

It follows that the right-hand side of (5.12) is dominated by the right-hand side of (5.6), if $h \in (0, h_0]$ and $h_0 > 0$ is sufficiently small.

Thus, in all situations

$$(\delta_{h,r} v_h)_+ \leq N(\bar{\mathcal{H}} + K + \|g\|_{C^{1,2}(\Omega_T)})$$

on Ω_T . Upon replacing here r with $-r$, we get

$$T_{h,-\ell_r}(\delta_{h,r} v_h)_- \leq N(\bar{\mathcal{H}} + K + \|g\|_{C^{1,2}(\Omega_T)})$$

in $(0, T) \times \Omega^h$, which after being combined with the previous estimate proves (5.6) in Ω^h . In $(0, T) \times (\Omega \setminus \Omega^h)$ estimate (5.6) has been established above.

Finally, we prove (5.7). This time denote

$$Q^o = \{(t, x) \in (0, T) \times [\Omega^h \cap \Lambda_\infty^h] : (\delta/4) \sum_k |\Delta_{h,k} v_h| > \bar{\mathcal{H}} + K\}.$$

Since v_h satisfies (5.3)-(5.4), estimate (5.7) obviously holds on $(\{T\} \times \Omega^h) \cup ((0, T) \times (\Omega \setminus \Omega^h))$. On $(0, T) \times [\Omega^h \cap \Lambda_\infty^h] \setminus Q^o$, we have

$$(\delta/4) \sum_k |\Delta_{h,k} v_h| \leq \bar{\mathcal{H}} + K,$$

which together with (5.5), (5.6), and (5.3) implies that (5.7) holds on $(0, T) \times [\Omega \cap \Lambda_\infty^h] \setminus Q^o$. Therefore, it remains to establish (5.7) on Q^o assuming that $Q^o \neq \emptyset$.

Observe that equation (5.10) holds on Q^o by the same reasons as above. Every x -section of Q^o is the union of open intervals on which $\partial_t v_h$ is Lipschitz continuous by virtue of (5.10). By subtracting the left-hand sides of (5.10)

evaluated at points t and $t + \varepsilon$, then transforming the difference by using Hadamard's formula, and finally dividing by ε and letting $\varepsilon \rightarrow 0$, we get that there exist functions a_k, b_k, c satisfying (5.8) such that on every x -section of Q^o (a.e.) we have

$$\partial_t(\partial_t v_h) + [a_k \Delta_{h,k} + b_k \delta_{h,k} - c](\partial_t v_h) = 0.$$

As above, owing to the continuity of $\partial_t v_h$ with respect to $t \in [0, T]$ and Lemma 4.2, we conclude

$$\sup_{Q^o} |\partial_t v_h| \leq \sup_{(0,T] \times [\Omega \cap \Lambda_\infty^h] \setminus Q^o} |\partial_t v_h|,$$

which implies (5.7) on Q^o . The lemma is proved. \square

Lemma 5.5. *Suppose that Assumptions 5.1 (i), (ii), (iv) are satisfied. Assume also that all vectors in Λ have rational coordinates. Then there are constants $h_0 > 0$ and N such that for all $h \in (0, h_0]$ and $|r| = 1, \dots, m$*

$$(\rho - 6\lambda h)|\Delta_{h,r} v_h| \leq N(\bar{\mathcal{H}} + K + \|g\|_{C^{1,2}(\Omega_T)}) \quad (5.13)$$

on $(0, T) \times \mathbb{R}^d$ (we remind the reader that λ is the radius of B).

Proof. As in the proof of Lemma 5.4 we will focus on proving (5.13) in $(0, T) \times \Lambda_\infty^h$. Then for a fixed r define

$$Q^o := \{(t, x) \in (0, T) \times [\Omega^{3h} \cap \Lambda_\infty^h] : (\delta/4)|\Delta_{h,r} v_h(t, x)| > \bar{\mathcal{H}} + K\}.$$

If $t \in (0, T)$, and $x \in \Lambda_\infty^h$ is such that $(t, x) \notin Q^o$, then either $x \notin \Omega^{3h}$, so that $\rho(x) \leq 3\lambda h$ and (5.13) holds, or else $x \in \Omega^{3h}$ but $(\delta/4)|\Delta_{h,r} v_h(t, x)| \leq \bar{\mathcal{H}} + K$, in which case (5.13) holds again.

Thus we need only prove (5.13) on Q^o assuming, of course, that $Q^o \neq \emptyset$. By Lemma 5.3 we have that (5.10) holds in Q^o and (5.11) holds in $Q \setminus Q^o$.

To proceed further observe a standard fact that there are constants $\mu_0 > 0$ and $N \in [0, \infty)$ depending only on Ω such that for any $\mu \in (0, \mu_0]$ there exists an $\eta_\mu \in C_0^\infty(\Omega)$ satisfying

$$\begin{aligned} \eta_\mu &= 1 \quad \text{on} \quad \Omega^{2\mu}, \quad \eta_\mu = 0 \quad \text{outside} \quad \Omega^\mu, \\ |\eta_\mu| &\leq 1, \quad |D\eta_\mu| \leq N/\mu, \quad |D^2\eta_\mu| \leq N/\mu^2. \end{aligned} \quad (5.14)$$

By Theorem 4.3 and Lemma 5.4 there are constants N and $h_0 > 0$ such that, for any number ν satisfying

$$\nu \geq N(\|\eta'_\mu\|_h + \|\eta''_\mu\|_h),$$

we have in Q^o that

$$\eta_\mu^4 [(\Delta_r v_h)^-]^2 \leq \sup_{Q \setminus Q^o} \eta_\mu^4 [(\Delta_r v_h)^-]^2 + N(\nu + 1)(\bar{\mathcal{H}} + K + \|g\|_{C^{1,2}(\Omega_T)})^2$$

if $h \in (0, h_0]$. We may certainly take h_0 smaller than $\mu_0/3$. In light of (5.14) one can take $\nu = N\mu^{-2}$ for an appropriate N and then

$$\eta_\mu^4 [(\Delta_r v_h(t, x))^-]^2 \leq \sup_{Q \setminus Q^o} \eta_\mu^4 [(\Delta_r v_h)^-]^2 + N\mu^{-2}(\bar{\mathcal{H}} + K + \|g\|_{C^{1,2}(\Omega_T)})^2$$

for $(t, x) \in Q^o$. While estimating the last supremum we will only concentrate on $\mu \in [3h, \mu_0]$ ($\neq \emptyset$), when $\eta_\mu = 0$ outside Ω^{3h} . In that case, for any $(s, y) \in Q \setminus Q^o$, either $y \notin \Omega^{3h}$ implying that

$$\eta_\mu^4[(\Delta_{h,r}v_h)^-]^2(s, y) = 0,$$

or $y \in \Omega^{3h} \cap \Lambda_\infty^h$ but

$$(\delta/4)|\Delta_r v_h(s, y)| \leq \bar{\mathcal{H}} + K, \quad (5.15)$$

or else $((s, y) \notin Q^o$ and) there is a sequence $s_n \uparrow s$ such that $(s_n, y) \in Q^o$.

The third possibility splits into two cases: 1) $s = T$, 2) $s < T$. In case 1 we have

$$|\Delta_r v_h(s, y)| = |\Delta_r g(s, y)| \leq N\|g\|_{C^{1,2}(\Omega_T)}.$$

In case 2, owing to the continuity of $\Delta_r v_h(t, y)$ with respect to t , estimate (5.15) holds again.

It follows that as long as $h \in (0, h_0]$, $(t, x) \in Q^o$, and $\mu \in [3h, \mu_0]$ we have

$$\eta_\mu^4[(\Delta_r v_h)^-(t, x)]^2 \leq N\mu^{-2}(\bar{\mathcal{H}} + K + \|g\|_{C^{1,2}(\Omega_T)})^2. \quad (5.16)$$

If x is such that $\rho(x) \geq 6\lambda h$, take $\mu = \mu_0 \wedge (\rho(x)/(2\lambda))$, which is bigger than $3h$ provided that $h \leq \mu_0/3$. In that case also $\rho(x) \geq 2\lambda\mu$, so that $\eta_\mu(x) = 1$ and we conclude from (5.16) that

$$\rho(x)(\Delta_r v_h)^-(t, x) \leq N(\bar{\mathcal{H}} + K + \|g\|_{C^{1,2}(\Omega_T)}),$$

$$(\rho(x) - 6\lambda h)(\Delta_r v_h)^-(t, x) \leq N(\bar{\mathcal{H}} + K + \|g\|_{C^{1,2}(\Omega_T)}) \quad (5.17)$$

for $(t, x) \in Q^o$ such that $\rho(x) \geq 6\lambda h$. However, the second relation in (5.17) is obvious for $\rho(x) \leq 6\lambda h$.

As a result of all the above arguments we see that (5.17) holds in $(0, T) \times \Lambda_\infty^h$ for any r whenever $h \in (0, h_0]$.

Finally, since $\partial_t v_h + P_h[v_h] \leq K$ in $(0, T) \times \Omega^h$, we have that

$$\begin{aligned} 2\delta^{-1} \sum_r (\Delta_r v_h)_+ &\leq -\partial_t v_h + (\delta/2) \sum_r (\Delta_r v_h)_- \\ &\quad - 2\delta^{-1} \sum_{r \geq 1} |\delta_r v_h| + (\delta/2)(v_h)_+ - 2\delta^{-1}(v_h)_- + K, \end{aligned}$$

which after being multiplied by $\rho - 6h$ along with (5.17) and Lemma 5.4 leads to (5.13) on $(0, T) \times \Lambda_\infty^h$. As is explained at the beginning of the proof, this finishes proving the lemma. \square

Mimicking the proof of Corollary 2.7 of [12], we obtain the following corollary from (5.6) and (5.13). Note that here Assumptions 5.1(iii) plays a crucial role and only the Lipschitz continuity in x is needed.

Corollary 5.6. *Suppose that Assumption 5.1 is satisfied and all vectors in Λ have rational coordinates. Then there are constants $h_0 > 0$ and M , which may depend on N' , such that for all $h \in (0, h_0]$, $t \in (0, T]$, and $x, y \in \Omega$, we have*

$$|v_h(t, x) - v_h(t, y)| \leq M(|x - y| + h).$$

Proof of Theorem 5.2. The theorem is proved in the same way as Theorem 8.10 of [10] on the basis of Lemmas 5.4 and 5.5 and the fact that the derivatives of v are weak limits of finite differences of v_h as $h \downarrow 0$. Thanks to Lemma 5.4, for each h sufficiently small and $x \in \Omega$, $v_h(t, x)$ are uniformly bounded and equicontinuous in $t \in [0, T]$. Let Q be the subset of Ω consisting of points with rational coordinates. By the Arzela–Ascoli theorem and Cantor’s diagonal argument, there is a sequence $h_n \rightarrow 0$ such that $v_{h_n}(t, x)$ converges uniformly on $[0, T] \times Q$. The limit function $v(t, x)$ satisfies

$$|v(t, x) - v(t, y)| \leq M|x - y| \quad (5.18)$$

for any $t \in [0, T]$ and $x, y \in Q$, where M is from Corollary 5.6. Since Q is dense in Ω , (5.18) allows us to extend v to $\bar{\Omega}_T$, with the extension denoted again by v being continuous in x . Note that $v(t, x)$ is Lipschitz in t with the Lipschitz constant bounded by the right-hand side of (5.7), which is independent on N' . Moreover, by (5.5) $v = g$ on $\partial'\Omega_T$ and $v_{h_n}(t, x)$ converges to $v(t, x)$ uniformly on Ω_T .

Next we estimate the second term on the left-hand side of (5.2). For any $\zeta \in C_0^\infty(\Omega_T)$ and for sufficiently small $h > 0$, from (5.6) we have

$$\left| \int_{Q_T} v_h \delta_{h,r} \zeta \, dx \, dt \right| = \left| \int_{Q_T} \delta_{h,-r} v_h \zeta \, dx \, dt \right| \leq N(\bar{\mathcal{H}} + K + \|g\|_{C^{1,2}(\Omega_T)}) \max_{Q_T} |\zeta|$$

for any $r = \pm 1, \dots, \pm m$, where N is independent of h . Passing to the limit as $h = h_n \rightarrow 0$, we obtain

$$\sup_{Q_T} |v(\ell_r)| \leq N(\bar{\mathcal{H}} + K + \|g\|_{C^{1,2}(\Omega_T)}).$$

Similarly, using (5.13) we get

$$\sup_{Q_T} |\rho v(\ell_r)(\ell_r)| \leq N(\bar{\mathcal{H}} + K + \|g\|_{C^{1,2}(\Omega_T)}). \quad (5.19)$$

Because $\pm e_i, \pm(e_i + e_j), e_i - e_j \in \Lambda$, $i, j = 1, \dots, d$, using the identity

$$2D_{ij}v = v_{(e_i+e_j)(e_i+e_j)} - v_{(\ell_i)(\ell_i)} - v_{(\ell_j)(\ell_j)},$$

we conclude from (5.19) that

$$\sup_{Q_T} |\rho D_{ij}v| \leq N(\bar{\mathcal{H}} + K + \|g\|_{C^{1,2}(\Omega_T)}).$$

This completes the proof of (5.2).

Finally, we show that v is a unique solution of (5.1) with the terminal-boundary condition $v = g$ on $\partial'\Omega_T$. Since $v \in W_{\infty, \text{loc}}^{1,2}(\Omega_T)$, at almost any point $(t_0, x_0) \in \Omega_T$ we have (see, for instance, Appendix 2 in [8])

$$v(t_0 + s, x_0 + y) = P^{t_0, x_0}(s, y) + o(|s| + |y|^2),$$

where

$$P^{t_0, x_0}(s, y) = v(t_0, x_0) + y^i D_i v(t_0, x_0) + \frac{1}{2} y^i y^j D_{ij} v(t_0, x_0) + s \partial_t v(t_0, x_0).$$

Take $\varepsilon > 0$ and observe that for all small $r > 0$, $|o(8r^2)| \leq 3\varepsilon r^2$, which implies that for

$$u(t, x) := P^{t_0, x_0}(t - t_0, x - x_0) + \varepsilon(t - t_0 + |x - x_0|^2 - r^2)$$

we have

$$u(t, x) \geq v(t, x) - |o(8r^2)| + 3\varepsilon r^2 \geq v(t, x)$$

on $\partial' D$, where

$$D = \{(t, y) : t_0 < t < t_0 + 4r^2, |y - x_0| < 2r\}.$$

We modify u outside D so that it is smooth with bounded derivatives in \mathbb{R}^{d+1} . It then follows from the comparison principle that for small enough h

$$v_h(t_0, x_0) - u(t_0, x_0) \leq \delta^{-1} \sup_D (\partial_t u + H_{K,h}[u])_+ + \sup_{\partial_h D \cup \partial' D} (v_h - u)_+, \quad (5.20)$$

where $\partial_h D = (t_0, t_0 + 4r^2) \times \{y : 2r - \lambda h \leq |y - x_0| \leq 2r\}$. Observe that $H_{K,h}[u] \rightarrow H_K[u]$ uniformly in D as $h \rightarrow 0$. Taking $h = h_n \rightarrow 0$ in (5.20), for sufficiently small $r > 0$ we have

$$\varepsilon r^2 = v(t_0, x_0) - u(t_0, x_0) \leq \delta^{-1} \sup_D (\partial_t u + H_K[u])_+.$$

It follows that for any sufficiently small $r > 0$, there is a point $(t_r, x_r) \in \bar{D}$ such that

$$\partial_t u(t_r, x_r) + H_K[u](t_r, x_r) > 0. \quad (5.21)$$

Note that

$$\begin{aligned} \partial_t u(t_r, x_r) &= \partial_t v(t_0, x_0) + \varepsilon, \\ \partial_{\ell_k} u(t_r, x_r) &= \partial_{\ell_k} v(t_0, x_0) + O(r), \\ \partial_{\ell_k}^2 u(t_r, x_r) &= \partial_{\ell_k}^2 v(t_0, x_0) + O(\varepsilon). \end{aligned}$$

Letting $r \rightarrow 0$ and then $\varepsilon \rightarrow 0$ in (5.21), we reach

$$\partial_t v(t_0, x_0) + H_K[v](t_0, x_0) \geq 0.$$

Similarly, we get an opposite inequality by considering

$$u(t, x) = P^{t_0, x_0}(t - t_0, x - x_0) - \varepsilon(t - t_0 + |x - x_0|^2 - r^2).$$

Therefore, $v \in \mathcal{C}^{1,1}(\bar{\Omega}_T) \cap W_{\infty, \text{loc}}^{1,2}(\Omega_T)$ is a solution to (5.1) with the terminal-boundary condition $v = g$ on $\partial' \Omega_T$. The uniqueness is a simple consequence of parabolic Alexandrov's estimate. The theorem is proved. \square

6. PROOF OF THEOREM 3.2

Here we suppose that the assumptions of Theorem 3.2 are satisfied and take the objects introduced in the end of Section 1. Owing to the assumptions of Theorem 3.2 by Theorem 7.1 of [10] (see the beginning of its proof in [10]) there exists a function $\mathcal{H}(z, t, x)$ defined for

$$z = (z', z''), \quad z' = (z'_0, \dots, z'_d) \in \mathbb{R}^{d+1}, \quad z'' \in \mathbb{R}^m, \quad (t, x) \in \mathbb{R}^{d+1}$$

such that:

(i) The function \mathcal{H} is Lipschitz continuous in z with Lipschitz constant $\hat{\delta}^{-1}$ and there exists a constant N' such that

$$|\mathcal{H}(z, t, x) - \mathcal{H}(z, s, y)| \leq N'(|t - s| + |x - y|)(1 + |z|)$$

for all $t, s \in \mathbb{R}$, $x, y \in \mathbb{R}^d$ and z .

(ii) We have $\mathcal{H}(z, t, x) = H(u, t, x)$ if $z' = u'$ and for all $j = 1, \dots, m$

$$z''_j = \langle u'' l_j, l_j \rangle.$$

In particular, $\mathcal{H}(0, t, x) = H(0, t, x)$ and if $v(t, x)$ is a real-valued function which is twice differentiable at a point $x \in \mathbb{R}^d$, at this point we have

$$H[v](t, x) = \mathcal{H}[v](t, x),$$

where

$$\mathcal{H}[v](t, x) = \mathcal{H}(v(t, x), Dv(t, x), v_{(l_1)(l_1)}(t, x), \dots, v_{(l_m)(l_m)}(t, x), t, x).$$

(iii) At all points (z, t, x) at which $\mathcal{H}(z, t, x)$ is differentiable with respect to z we have

$$|\mathcal{H}_{z'_i}(z, t, x)| \leq 4\delta^{-1}, \quad i = 1, \dots, d, \quad (6.1)$$

$$\delta/4 \leq -\mathcal{H}_{z'_0}(z, t, x) \leq 4\delta^{-1}, \quad \hat{\delta}^{-1} \geq \mathcal{H}_{z''_j}(z, t, x) \geq \hat{\delta}, \quad j = 1, \dots, m. \quad (6.2)$$

The proofs in [10] use the fact that (3.9) holds and yield the function \mathcal{H} such that, in addition, at all points (z, t, x) at which $\mathcal{H}(z, t, x)$ is differentiable with respect to z we also have

$$|\mathcal{H}(z, t, x) - \langle z, D_z \mathcal{H}(z, t, x) \rangle| \leq 2N_0.$$

However, the latter property of \mathcal{H} will not be used here, so that we only used assumption (3.9) to be sure that \mathcal{H} with the properties (i)-(iii) exists.

The functions \mathcal{H} from above and \mathcal{P} from Section 1 are instances of \mathcal{H} and \mathcal{P} from Section 5. To see this, of course, one has to change the constant δ in Section 5 and renumber the l_i 's in Section 1. We also take into account that $\hat{\delta} \leq \delta/4$ which allows us to match (6.1) and (6.2) with the requirements of Assumption 5.1 (i). Furthermore, $\bar{\mathcal{H}} = \bar{H}$. Therefore, Theorem 5.2 is applicable and yields a unique solution $v \in \mathcal{C}^{1,1}(\bar{\Omega}_T) \cap W_{\infty, \text{loc}}^{1,2}(\Omega_T)$ such that $v = g$ on $\partial' \Omega_T$, estimates (5.2), that is (1.3), hold true, and

$$\begin{aligned} \partial v_t + \max[\mathcal{H}(v, Dv, v_{(l_1)(l_1)}, \dots, v_{(l_m)(l_m)}, t, x), \\ \mathcal{P}(v, Dv, v_{(l_1)(l_1)}, \dots, v_{(l_m)(l_m)}) - K] = 0 \end{aligned}$$

in Ω_T (a.s.). In light of the construction of \mathcal{H} this equation coincides with (1.2), so that the only remaining assertions of Theorem 3.2 to prove are that for $p > d + 1$

$$\|v\|_{W_p^{1,2}(\Omega_T)} \leq N_p(\bar{H} + K + \|g\|_{W_p^{1,2}(\Omega_T)}) \quad (6.3)$$

and estimate (1.5) holds. The latter follows from other assertions of Theorem 3.2 by Remark 1.4, so that we may concentrate on (6.3).

Observe that

$$\partial u_t + \max(H(u, t, x), P(u) - K) = \partial u_t + P(u) + G(u, t, x),$$

where

$$G(u, t, x) = (H(u, t, x) - P(u) + K)_+ - K$$

and, owing to condition (3.10), $G(u, x) = -K$ if

$$\kappa\left(\sum_{i,j} |u_{ij}| + \sum_i |u_i|\right) \geq \bar{H} + K.$$

If the opposite inequality holds, then

$$|G(u, t, x)| \leq |H(u, t, x) - H(0, t, x)| + |P(u)| + \bar{H} + K \leq N(\bar{H} + K), \quad (6.4)$$

where N depends only on δ and d . It follows that the inequality between the extreme terms in (6.4) holds for all u and (t, x) . This allows us to apply Theorem 1.2 of [6] and shows that (6.3) holds if $v \in W_p^{1,2}(\Omega_T)$. Since P is convex with respect to u'' and $G(v, t, x)$ is bounded, due to Theorem 1.2 of [6] there is a unique solution $w \in W_p^{1,2}(\Omega_T)$ to the equation $\partial w_t + P(w) = -G(v, t, x)$ with the terminal-boundary condition $w = g$ on $\partial'\Omega_T$. By uniqueness of $W_{d+1,\text{loc}}^{1,2}(\Omega_T) \cap C(\bar{\Omega}_T)$ -solutions we obtain $w = v \in W_p^{1,2}(\Omega_T)$ and the theorem is proved.

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